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An optimal double inequality between geometric and identric means

Miao-Kun Wang [a], Zi-Kui Wang [b], Yu-Ming Chu [a,*]

[a] Department of Mathematics, Huzhou Teachers College, Huzhou 313000, China

[b] Department of Mathematics, Hangzhou Normal University, Hangzhou 310018, China

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ABSTRACT

We find the greatest value p and least value q in $(0, 1/2)$ such that the double inequality $G(pa+(1-p)b, pb+(1-p)a) < I(a,b) < G(qa+(1-q)b, qb+(1-q)a)$ holds for all $a, b > 0$ with $a \neq b$. Here, $G(a,b)$, and $I(a,b)$ denote the geometric, and identric means of two positive numbers a and b, respectively.

1. Introduction

The classical geometric mean $G(a,b)$ and identric mean $I(a,b)$ of two positive numbers a and b with $a \neq b$ are defined by

$$G(a,b) = \sqrt{ab} \quad \text{and} \quad I(a,b) = \frac{1}{e}\left(b^b/a^a\right)^{1/(b-a)},$$

respectively. Recently, both mean values have been the subject of intensive research. In particular, many remarkable inequalities for G and I can be found in the literature [1–21].

Let $M_p(a,b) = [(a^p+b^p)/2]^{1/p} (p \neq 0)$ and $M_0(a,b) = \sqrt{ab}$, $H(a,b) = 2ab/(a+b)$, $A(a,b) = (a+b)/2$, $L(a,b) = (a-b)/(\log a - \log b)$, and $P(a,b) = (a-b)/(4\arctan\sqrt{a/b} - \pi)$ be the p-th power, harmonic, arithmetic, logarithmic, and Seiffert means of two positive numbers a and b with $a \neq b$, respectively. Then it is well known that

$$\begin{aligned} &\min\{a,b\} < H(a,b) = M_{-1}(a,b) < G(a,b) = M_0(a,b) < L(a,b) \\ &\quad < P(a,b) < I(a,b) < A(a,b) = M_1(a,b) < \max\{a,b\} \end{aligned}$$

for all $a, b > 0$ with $a \neq b$.

The following sharp bounds for I, $(LI)^{1/2}$, and $(L+I)/2$ in terms of power mean are presented in [22].

$$\begin{aligned} &M_{2/3}(a,b) < I(a,b) < M_{\log 2}(a,b), \qquad M_0(a,b) < \sqrt{L(a,b)I(a,b)} < M_{1/2}(a,b), \\ &M_{\log 2/(1+\log 2)}(a,b) < [L(a,b)+I(a,b)]/2 < M_{1/2}(a,b) \end{aligned}$$

for all $a, b > 0$ with $a \neq b$.

In [23], Alzer and Qiu proved that the inequalities

$$\alpha A(a,b) + (1-\alpha)G(a,b) < I(a,b) < \beta A(a,b) + (1-\beta)G(a,b) \tag{1.1}$$

hold for all positive real numbers a and b with $a \neq b$ if and only if $\alpha \leq 2/3$ and $\beta \geq 2/e = 0.73575\cdots$.

* Corresponding author.
E-mail address: chuyuming@hutc.zj.cn (Y.-M. Chu).

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In [24–26], the authors answered the questions: for $\alpha, \beta \in (0, 1)$ with $\alpha + \beta < 1$, what are the greatest values p_1, p_2, p_3 and the least values q_1, q_2, q_3 such that $M_{p_1}(a, b) < P^{\alpha}(a, b)G^{1-\alpha}(a, b) < M_{q_1}(a, b)$, $M_{p_2}(a, b) < G^{\alpha}(a, b)L^{1-\alpha}(a, b) < M_{q_2}(a, b)$, and $M_{p_3}(a, b) < A^{\alpha}(a, b)G^{\beta}(a, b)H^{1-\alpha-\beta}(a, b) < M_{q_3}(a, b)$ for all $a, b > 0$ with $a \neq b$?

For fixed $a, b > 0$ with $a \neq b$ and $x \in [0, 1/2]$, let

$$g(x) = G\left(xa + (1-x)b, xb + (1-x)a\right).$$

Then it is not difficult to verify that $g(x)$ is continuous and strictly increasing in $[0, 1/2]$. Note that $g(0) = G(a, b) < I(a, b)$ and $g(1/2) = A(a, b) > I(a, b)$. Therefore, it is natural to ask what are the greatest value p and least value q in $(0, 1/2)$ such that the double inequality $G(pa + (1-p)b, pb + (1-p)a) < I(a, b) < G(qa + (1-q)b, qb + (1-q)a)$ holds for all $a, b > 0$ with $a \neq b$. The main purpose of this paper is to answer this question. Our main result is the following Theorem 1.1.

Theorem 1.1. *If $p, q \in (0, 1/2)$, then the double inequality*

$$G\left(pa + (1-p)b, pb + (1-p)a\right) < I(a, b) < G\left(qa + (1-q)b, qb + (1-q)a\right) \tag{1.2}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $p \leq \left(1 - \sqrt{1 - 4/e^2}\right)/2$ and $q \geq \left(3 - \sqrt{3}\right)/6$.

2. Proof of Theorem 1.1

Proof of Theorem 1.1. Let $\lambda = (3 - \sqrt{3})/6$ and $\mu = \left(1 - \sqrt{1 - 4/e^2}\right)/2$. Then from the monotonicity of the function $g(x) = G\left(xa + (1-x)b, xb + (1-x)a\right)$ in $[0, 1/2]$ we know that to prove inequality (1.2) we only need to prove that inequalities

$$I(a, b) < G\left(\lambda a + (1-\lambda)b, \lambda b + (1-\lambda)a\right) \tag{2.1}$$

and

$$I(a, b) > G\left(\mu a + (1-\mu)b, \mu b + (1-\mu)a\right) \tag{2.2}$$

hold for all $a, b > 0$ with $a \neq b$.

Without loss of generality, we assume that $a > b$. Let $t = a/b > 1$ and $r \in (0, 1/2)$, then simple computation leads to

$$\begin{aligned}
&\log G\left(ra + (1-r)b, rb + (1-r)a\right) - \log I(a, b) \\
&= \frac{1}{2}\log\left(\frac{ra}{b} + 1 - r\right) + \frac{1}{2}\log\left[\frac{(1-r)a}{b} + r\right] + \log b - \frac{a\left(\log a - \log b\right)}{a - b} - \log b + 1 \\
&= \frac{1}{2}\log\left(rt + 1 - r\right) + \frac{1}{2}\log\left[(1-r)t + r\right] - \frac{t}{t-1}\log t + 1.
\end{aligned} \tag{2.3}$$

Let

$$f(t) = \frac{1}{2}\log\left(rt + 1 - r\right) + \frac{1}{2}\log\left[(1-r)t + r\right] - \frac{t}{t-1}\log t + 1, \tag{2.4}$$

then

$$f(1) = 0, \tag{2.5}$$

$$\lim_{t \to +\infty} f(t) = \frac{1}{2}\log\left[r(1-r)\right] + 1, \tag{2.6}$$

$$f'(t) = \frac{f_1(t)}{2(t-1)^2}, \tag{2.7}$$

where

$$f_1(t) = 2\log t - \frac{t^2 - 1}{\left[rt + (1-r)\right]\left[(1-r)t + r\right]},$$

$$f_1(1) = 0, \tag{2.8}$$

$$\lim_{t \to +\infty} f_1(t) = +\infty, \tag{2.9}$$

$$f_1'(t) = \frac{f_2(t)}{t\left[rt + (1-r)\right]^2\left[(1-r)t + r\right]^2}, \tag{2.10}$$

where

$$f_2(t) = 2r^2(1-r)^2t^4 - (2r^2-2r+1)(1-2r)^2t^3 + 2(6r^4-12r^3+12r^2 - 6r+1)t^2 - (2r^2-2r+1)(1-2r)^2t + 2r^2(1-r)^2,$$

$$f_2(1) = 0, \tag{2.11}$$

$$\lim_{t\to+\infty} f_2(t) = +\infty, \tag{2.12}$$

$$f_2'(t) = 8r^2(1-r)^2t^3 - 3(2r^2-2r+1)(1-2r)^2t^2 + 4(6r^4-12r^3 + 12r^2-6r+1)t - (2r^2-2r+1)(1-2r)^2,$$

$$f_2'(1) = 0, \qquad \lim_{t\to+\infty} f_2'(t) = +\infty, \tag{2.13}$$

$$f_2''(t) = 24r^2(1-r)^2t^2 - 6(2r^2-2r+1)(1-2r)^2t + 4(6r^4-12r^3+12r^2-6r+1),$$

$$f_2''(1) = -2(6r^2-6r+1), \tag{2.14}$$

$$\lim_{t\to+\infty} f_2''(t) = +\infty, \tag{2.15}$$

$$f_2'''(t) = 48r^2(1-r)^2t - 6(2r^2-2r+1)(1-2r)^2, \tag{2.16}$$

$$f_2'''(1) = -6(6r^2-6r+1) \tag{2.17}$$

and

$$\lim_{t\to+\infty} f_2'''(t) = +\infty. \tag{2.18}$$

We divide the proof into two cases.

Case 1. $r = \lambda = (3-\sqrt{3})/6$. Then Eqs. (2.14) and (2.17) lead to

$$f_2''(1) = 0 \tag{2.19}$$

and

$$f_2'''(1) = 0. \tag{2.20}$$

It follows from (2.16) that $f_2'''(t)$ is strictly increasing in $[1, +\infty)$, then (2.20) implies that $f_2'''(t) > 0$ for $t \in (1, +\infty)$. Hence, $f_2''(t)$ is strictly increasing in $[1, +\infty)$.

From Eqs. (2.5), (2.7), (2.8), (2.10), (2.11), (2.13) and (2.19) together with the monotonicity of $f_2''(t)$ we clearly see that

$$f(t) > 0 \tag{2.21}$$

for $t \in (1, +\infty)$.

Therefore, inequality (2.1) follows from Eqs. (2.3) and (2.4) together with inequality (2.21).

Case 2. $r = \mu = (1-\sqrt{1-4/e^2})/2$. Then Eqs. (2.14) and (2.17) lead to

$$f_2''(1) = 2\left(\frac{6}{e^2}-1\right) < 0 \tag{2.22}$$

and

$$f_2'''(1) = 6\left(\frac{6}{e^2}-1\right) < 0. \tag{2.23}$$

It follows from (2.18) and (2.23) together with the monotonicity of $f_2'''(t)$ that there exists $t_1 > 1$ such that $f_2'''(t) < 0$ for $t \in (1, t_1)$ and $f_2'''(t) > 0$ for $t \in (t_1, +\infty)$. Hence, $f_2''(t)$ is strictly decreasing in $[1, t_1]$ and strictly increasing in $[t_1, +\infty)$.

From (2.15) and (2.22) together with the piecewise monotonicity of $f_2''(t)$ we clearly see that there exists $t_2 > t_1 > 1$ such that $f_2'(t)$ is strictly decreasing in $[1, t_2]$ and strictly increasing in $[t_2, +\infty)$. Then (2.13) leads to the conclusion that there exists $t_3 > t_2 > 1$ such that $f_2(t)$ is strictly decreasing in $[1, t_3]$ and strictly increasing in $[t_3, +\infty)$.

It follows from (2.10) to (2.12) and the piecewise monotonicity of $f_2(t)$ that there exists $t_4 > t_3 > 1$ such that $f_1(t)$ is strictly decreasing in $[1, t_4]$ and strictly increasing in $[t_4, +\infty)$.

From (2.7) to (2.9) and the piecewise monotonicity of $f_1(t)$ we know that there exists $t_5 > t_4 > 1$ such that $f(t)$ is strictly decreasing in $[1, t_5]$ and strictly increasing in $[t_5, +\infty)$.

Note that Eq. (2.6) becomes

$$\lim_{t\to+\infty} f(t) = 0 \tag{2.24}$$

for $r = \mu = (1-\sqrt{1-4/e^2})/2$.

From (2.5) and (2.24) together with the piecewise monotonicity of $f(t)$ we conclude that

$$f(t) < 0 \tag{2.25}$$

for $t > 1$.

Therefore, inequality (2.2) follows from Eqs. (2.3) and (2.4) together with inequality (2.25).

Next, we prove that $\lambda = (3-\sqrt{3})/6$ is the best possible parameter in $(0, 1/2)$ such that inequality (2.1) holds for all $a, b > 0$ with $a \neq b$. In fact, if $0 < r < \lambda = (3-\sqrt{3})/6$, then (2.14) leads to $f_2''(1) = -2(6r^2 - 6r + 1) < 0$. Thus from the continuity of $f_2''(t)$ we know that there exists $\delta > 0$ such that

$$f_2''(t) < 0 \tag{2.26}$$

for $t \in (1, 1+\delta)$.

It follows from (2.3) to (2.5), (2.7), (2.8), (2.10), (2.11), (2.13) and (2.26) that $I(a, b) > G(ra + (1-r)b, rb + (1-r)a)$ for $a/b \in (1, 1+\delta)$.

Finally, we prove that the parameter $\mu = (1-\sqrt{1-4/e^2})/2$ is the best possible parameter in $(0, 1/2)$ such that inequality (2.2) holds for all $a, b > 0$ with $a \neq b$. In fact, if $(1-\sqrt{1-4/e^2})/2 = \mu < r < 1/2$, then (2.6) leads to the conclusion that $\lim_{t\to+\infty} f(t) > 0$. Hence there exists $T > 1$ such that

$$f(t) > 0 \tag{2.27}$$

for $t \in (T, +\infty)$.

Therefore, $I(a, b) < G(ra + (1-r)b, rb + (1-r)a)$ for $a/b \in (T, +\infty)$ follows from Eqs. (2.3) and (2.4) together with inequality (2.27). $\square$

Remark 2.1. The lower and upper bounds in (1.2) with $p = (1-\sqrt{1-4/e^2})/2$ and $q = (3-\sqrt{3})/6$ are better than those in (1.1) with $\alpha = 2/3$ and $\beta = 2/e$ for some $a, b > 0$, respectively. In fact, if we let $t = \sqrt{a/b} > 0$, then

$$\begin{aligned}
&[G(pa + (1-p)b, pb + (1-p)a)]^2 - [\alpha A(a, b) + (1-\alpha)G(a, b)]^2 \\
&= [pa + (1-p)b]\,[pb + (1-p)a] - \left[\frac{a+b}{3} + \frac{\sqrt{ab}}{3}\right]^2 \\
&= \frac{b^2(t-1)^2}{9e^2}\left[(9-e^2)t^2 + 2(9-2e^2)t + 9 - e^2\right] \\
&= \frac{b^2(9-e^2)(t-1)^2}{9e^2}\left[t - \frac{2e^2 - 9 - \sqrt{3e^2(e^2-6)}}{9-e^2}\right]\left[t - \frac{2e^2 - 9 + \sqrt{3e^2(e^2-6)}}{9-e^2}\right]
\end{aligned} \tag{2.28}$$

and

$$\begin{aligned}
&[G(qa + (1-q)b, qb + (1-q)a)]^2 - [\beta A(a, b) + (1-\beta)G(a, b)]^2 \\
&= [qa + (1-q)b]\,[qb + (1-q)a] - \left[\frac{a+b}{e} + \frac{(e-2)\sqrt{ab}}{e}\right]^2 \\
&= \frac{b^2(t-1)^2}{6e^2}\left[(e^2-6)t^2 + 2(e^2 - 6e + 6)t + e^2 - 6\right] \\
&= \frac{b^2(e^2-6)(t-1)^2}{6e^2}\left[t - \frac{-e^2 + 6e - 6 - \sqrt{12e(e-2)(3-e)}}{e^2-6}\right] \\
&\quad \times \left[t - \frac{-e^2 + 6e - 6 + \sqrt{12e(e-2)(3-e)}}{e^2-6}\right].
\end{aligned} \tag{2.29}$$

Eqs. (2.28) and (2.29) imply that $G(pa + (1-p)b, pb + (1-p)a) > 2A(a, b)/3 + G(a, b)/3$ if and only if $a/b < \left[2e^2 - 9 - \sqrt{3e^2(e^2-6)}\right]^2/(9-e^2)^2 = 0.020\cdots$ or $a/b > \left[2e^2 - 9 + \sqrt{3e^2(e^2-6)}\right]^2/(9-e^2)^2 = 49.439\cdots$, and $G(qa + (1-q)b, qb + (1-q)a) < 2A(a, b)/e + (e-2)G(a, b)/e$ if and only if $\left[-e^2 + 6e - 6 - \sqrt{12e(e-2)(3-e)}\right]^2/(e^2-6)^2 = 0.064\cdots < a/b < \left[-e^2 + 6e - 6 + \sqrt{12e(e-2)(3-e)}\right]^2/(e^2-6)^2 = 15.619\cdots$. $\square$

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